Iterative Solvers for Large Linear Systems

Part IIa: Jacobi Method

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Outline

- Basics of Iterative Methods
- Splitting-schemes
  - Jacobi- u. Gauß-Seidel-scheme
  - Relaxation methods
- Methods for symmetric, positive definite Matrices
  - Method of steepest descent
  - Method of conjugate directions
  - CG-scheme

Outline

- Multigrid Method
  - Smoother, Prolongation, Restriction
  - Twogrid Method and Extension
- Methods for non-singular Matrices
  - GMRES
  - BiCG, CGS and BiCGSTAB
- Preconditioning
  - ILU, IC, GS, SGS, ...

Jacobi method

<u>Procedure:</u> Write $A = D + L + R$ by means of

- $D = diag\{a_{11}, \ldots, a_{nn}\}$
- $L = \begin{pmatrix} 0 & \cdot & \cdot & 0 \\ a_{21} & \cdot & & \cdot \\ \vdots & \ddots & \cdot & \cdot \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \ddots & \vdots \\ \cdot & & \cdot & a_{n-1,n} \\ 0 & \cdot & \cdot & 0 \end{pmatrix}$
- Choose $B_J = D$

$$\Longrightarrow \quad M_J = B_J^{-1}(B_J - A) = -D^{-1}(L + R), \ \ N_J = B_J^{-1} = D^{-1}$$

$$\Longrightarrow \quad x_{m+1} = -D^{-1}(L + R)x_m + D^{-1}b$$

- $$x_{m+1,i} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_{m,j} \right), \ i = 1, \ldots, n$$

Jacobi method

$$x_{m+1,i} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_{m,j} \right), \ i = 1, \ldots, n$$

Properties:

- Calculation of $x_{m+1,i}$ by means of the vector

$$(x_{m,1}, \ldots, x_{m,i-1}, 0, x_{m,i+1}, \ldots, x_{m,n})^T$$

- Independent of the numbering of the unknowns
- Very well suited for parallel computing

Appraisement:

"$\circ$" : Minor assumptions on the matrix A, $(a_{ii} \neq 0$ for $i = 1, \ldots, n)$

"$+$" : Very simple calculation of matrix-vector products $B_J^{-1} x = D^{-1} x$

"$\circ$" : Moderate approximation of A $(B_J \longleftrightarrow A)$

Convergence of the Jacobi method

First idea:

By means of the calculation of

$$\|M\| < 1$$

one directly obtains convergence due to

$$\rho(M) \leq \|M\| < 1.$$

Utilizing

$$M_J = D^{-1}(D - A) = -\begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \dots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \ddots & \vdots \\ \vdots & & \ddots & \frac{a_{n-1,n}}{a_{n-1,n-1}} \\ \frac{a_{n1}}{a_{nn}} & \dots & \frac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix}$$

we recognize:

Jacobi method

Convergence criteria

If the matrix A with $a_{ii} \neq 0 \quad , i = 1, \dots, n$ fulfills

$$q_\infty := \max_{i=1,\dots,n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{|a_{ik}|}{|a_{ii}|} < 1 \qquad \text{Strict row diagonal dominance}$$

or

$$q_1 := \max_{k=1,\dots,n} \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{|a_{ik}|}{|a_{ii}|} < 1 \qquad \text{Strict column diagonal dominance}$$

or

$$q_2 := \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(\frac{|a_{ik}|}{|a_{ii}|} \right)^2 < 1,$$

then the Jacobi scheme will converge to the solution vector $x^\star = A^{-1}b$ independent of the right hand side b as well as the initial guess x_0.

Model problem

$$Ax = b \;\; with \;\; A = \begin{pmatrix} 0.7 & -0.4 \\ -0.2 & 0.5 \end{pmatrix}, \quad b = \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}$$

- Convergence criterion

$$M_J = -D^{-1}(D - A) = \begin{pmatrix} 0 & 4/7 \\ 2/5 & 0 \end{pmatrix} \Longrightarrow q_\infty = \max\left\{\tfrac{4}{7}, \tfrac{2}{5}\right\} = \tfrac{4}{7} < 1$$

$\Longrightarrow$ Jacobi scheme is convergent

- Rate of convergence: Eigenvalues of the iteration matrix M_J

$$0 = \det(M_J - \lambda I) = \lambda^2 - \frac{8}{35} \quad \Longrightarrow \quad \lambda_{1,2} = \pm\sqrt{\frac{8}{35}}$$

$$\rho(M_J) = \sqrt{\frac{8}{35}} \approx 0.478 \approx 0.7^2$$

Expectation:

Approximately half as much iterations as the trivial method to reach the same accuracy

Jacobi method

Jacobi method				
m	$x_{m,1}$	$x_{m,2}$	$\varepsilon_m := \|x_m - x^*\|_\infty$	$\varepsilon_m/\varepsilon_{m-1}$
0	2.100000e+01	-1.900000e+01	2.000000e+01	
1	-1.042857e+01	9.000000e+00	1.142857e+01	5.714286e-01
2	5.571429e+00	-3.571429e+00	4.571429e+00	4.000000e-01
3	-1.612245e+00	2.828571e+00	2.612245e+00	5.714286e-01
4	2.044898e+00	-4.489796e-02	1.044898e+00	4.000000e-01
5	4.029155e-01	1.417959e+00	5.970845e-01	5.714286e-01
6	1.238834e+00	7.611662e-01	2.388338e-01	4.000000e-01
7	8.635235e-01	1.095534e+00	1.364765e-01	5.714286e-01
8	1.054591e+00	9.454094e-01	5.459059e-02	4.000000e-01
9	9.688054e-01	1.021836e+00	3.119462e-02	5.714286e-01
10	1.012478e+00	9.875222e-01	1.247785e-02	4.000000e-01
11	9.928698e-01	1.004991e+00	7.130199e-03	5.714286e-01
12	1.002852e+00	9.971479e-01	2.852080e-03	4.000000e-01
13	9.983702e-01	1.001141e+00	1.629760e-03	5.714286e-01
14	1.000652e+00	9.993481e-01	6.519039e-04	4.000000e-01
15	9.996275e-01	1.000261e+00	3.725165e-04	5.714286e-01
20	1.000008e+00	9.999922e-01	7.784835e-06	4.000000e-01
25	9.999998e-01	1.000000e+00	2.324102e-07	5.714286e-01
30	1.000000e+00	1.000000e-00	4.856900e-09	4.000000e-01
35	1.000000e-00	1.000000e+00	1.449989e-10	5.714279e-01
40	1.000000e+00	1.000000e-00	3.030243e-12	4.000117e-01
45	1.000000e-00	1.000000e+00	9.037215e-14	5.700280e-01
48	1.000000e+00	1.000000e-00	8.437695e-15	4.086022e-01

Jacobi method

Model problem:

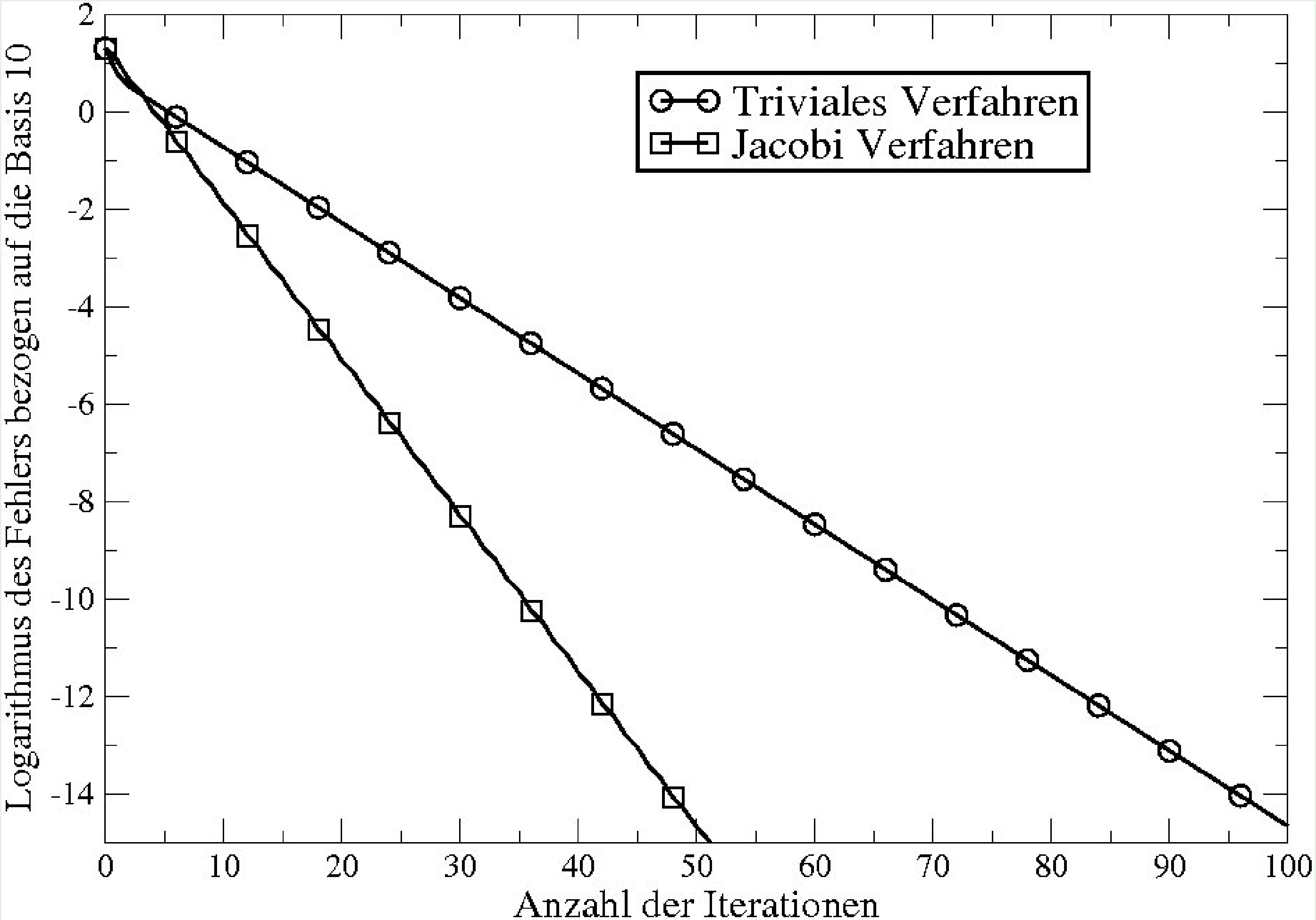

Abbildung: Convergence history $\log_{10} \varepsilon_m$ of the Jacobi method

Example: 1-D Poisson equation

A central scheme yields the linear system of equations:

$$Ax = b$$

where

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{pmatrix}$$

For $n > 2$ one obtains $q_\infty = q_1 = 1$ and $q_2 \geq 1$.

Furthermore,

$$\left.\begin{array}{l} A \text{ is irreducible} \\ \displaystyle\sum_{j=2}^{n} \frac{|a_{1j}|}{|a_{11}|} = \frac{1}{2} < 1 \end{array}\right\} \Longrightarrow \text{Jacobi scheme is convergent.}$$

Jacobi method

Second idea (Convergence):

- Acceptance of the property $q_\infty = 1$ in combination with an additional requirement.

Convergence criterion

Let A be irreducible with

$$q_\infty \leq 1.$$

If there exists an index $k \in \{1, \dots, n\}$ such that

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{|a_{kj}|}{|a_{kk}|} < 1,$$

then the Jacobi scheme will converge to the solution vector $x^\star = A^{-1}b$ independent of the right hand side b as well as the initial guess x_0.

Jacobi method

Definition:Irreducibility

A matrix is called reducible, if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}$$

where $\tilde{A}_{ii} \in \mathbb{C}^{n_i \times n_i}, \quad n_1 + n_2 = n$.
Otherwise A is called irreducible.

Irreducibility means:

A matrix is irreducible, if for each pair of indices (k, i) there exists a directed path of length $m + 1$

$$(k, k_1)(k_1, k_2) \dots (k_m, l).$$

Thereby a path (i,j) exists if and only if$a_{ij} \neq 0$.

Jacobi method

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A matrix is irreducible, if for each pair of indices (k, l) there exists a directed path of length $m+1$

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Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ reducible} \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ irreducible}$$

Effect of the irreducibility:

A reduction of the error in the k-th row will successively leads to a reduction of the error in each row $\Longrightarrow$ Convergence