Iterative Solvers for Large Linear Systems
Part IIb: Gauß-Seidel Method

Andreas Meister

University of Kassel, Department of Analysis and Applied Mathematics

Outline

- Basics of Iterative Methods
- Splitting-schemes
  - Jacobi- u. Gauß-Seidel-scheme
  - Relaxation methods
- Methods for symmetric, positive definite Matrices
  - Method of steepest descent
  - Method of conjugate directions
  - CG-scheme

Outline

- Multigrid Method
  - Smoother, Prolongation, Restriction
  - Twogrid Method and Extension
- Methods for non-singular Matrices
  - GMRES
  - BiCG, CGS and BiCGSTAB
- Preconditioning
  - ILU, IC, GS, SGS, ...

Gauß-Seidel method

Procedure: Write $A = D + L + R$

- Choose $B_{GS} = D + L$

$$\Longrightarrow \quad M_{GS} = (D+L)^{-1}(D+L-A) = -(D+L)^{-1}R,$$

$$N_{GS} = (D+L)^{-1}$$

$$\Longrightarrow \quad x_{m+1} = -(D+L)^{-1}Rx_m + (D+L)^{-1}b$$

- Problem :
  - Calculation of $(D+L)^{-1}$ may be expensive
  - $(D+L)^{-1}$ may be a full matrix
    (PDEs often lead to sparse matrices)
- Solution
  - Component-by-component derivation

Gauß-Seidel method

Transformation:
$$x_{m+1} = -(D+L)^{-1}Rx_m + (D+L)^{-1}b$$
$$(D+L)x_{m+1} = -Rx_m + b$$

Consider the i-th component:

$$\sum_{j=1}^{i} a_{ij}x_{m+1,j} = -\sum_{j=i+1}^{n} a_{ij}x_{m,j} + b_i$$
$$\Longrightarrow x_{m+1,i} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_{m+1,j} - \sum_{j=i+1}^{n} a_{ij}x_{m,j}\right), \; i = 1,\ldots,n$$

Properties:

- Calculation of $x_{m+1,i}$ by means of
  $$(x_{m+1,1},\ldots,x_{m+1,i-1},0,x_{m,i+1},\ldots,x_{m,n})^T.$$
- Dependent on the numbering of the unknowns.
- Horrible for parallel computing

Gauß-Seidel method

Formulation using matrices:

$$x_{m+1} = -(D+L)^{-1} R x_m + (D+L)^{-1} b$$

Pointwise counterpart:

$$x_{m+1,i} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_{m+1,j} - \sum_{j=i+1}^{n} a_{ij} x_{m,j} \right), \quad i = 1, \ldots, n$$

Appraisement:

$\circ$: Minor assumptions on A
($a_{ii} \neq 0 \quad$ für $\quad i = 1, \ldots, n$)

$+$: Simple calculation of matrix-vector-products by $B_{GS}^{-1} x = (D+L)^{-1} x$

$+$: Good approximation of A
($B_{GS} \longleftrightarrow A$)

Gauß-Seidel method

Convergence criterion

Let A be a square matrix with diagonal elements $a_{ii} \neq 0$. If the auxiliary quantities

$$p_i = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} p_j + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \quad , i = 1, \ldots, n$$

satisfy

$$p = \max_{i=1,\ldots,n} p_i < 1,$$

then the Gauß-Seidel scheme will converge to the solution vector $x^\star = A^{-1}b$ independent of the right hand side b as well as the initial guess x_0.

Main idea of the proof:

The constraint yields $\|M_{GS}\|_\infty < 1$ that proofs the convergence due to $\rho(M_{GS}) \leq \|M_{GS}\|_\infty$.

Gauß-Seidel method

Example:

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$$p_1 = \tfrac{1}{2} < 1$$

$$p_i = \tfrac{1}{2}p_{i-1} + \tfrac{1}{2} < 1, \quad i = 2, \ldots, n$$

$$p_n = \tfrac{1}{2}p_{n-1} < 1$$

$\Longrightarrow$ Gauß-Seidel method is convergent.

Modell problem

$$Ax = b \text{ with } A = \begin{pmatrix} 0.7 & -0.4 \\ -0.2 & 0.5 \end{pmatrix}, \quad b = \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}$$

- Convergence criterion

$$p_1 = \tfrac{4}{7}, \quad p_2 = \tfrac{2}{5} \cdot \tfrac{4}{7} = \tfrac{8}{35} \qquad \Longrightarrow \qquad p = \max\{p_1, p_2\} < 1$$

$\Longrightarrow$ Gauß-Seidel method is convergent

- Rate of convergence: Eigenvalues of the iteration matrix M_{GS}

$$0 = \det(M_{GS} - \lambda I) = \det \begin{pmatrix} -\lambda & 4/7 \\ 0 & -8/35 - \lambda \end{pmatrix} \Longrightarrow \lambda_1 = 0, \ \lambda_2 = -\frac{8}{35}$$

$$\rho(M_{GS}) = \frac{8}{35} = \rho(M_J)^2 \approx 0.22857$$

Expectation:

Approximately two times faster convergence compared to the Jacobi scheme

Gauß-Seidel method

Gauß-Seidel method				
m	$x_{m,1}$	$x_{m,2}$	$\varepsilon_m := \|x_m - x^*\|_\infty$	$\varepsilon_m/\varepsilon_{m-1}$
0	2.100000e+01	-1.900000e+01	2.000000e+01	
1	-1.042857e+01	-3.571429e+00	1.142857e+01	5.714286e-01
2	-1.612245e+00	-4.489796e-02	2.612245e+00	2.285714e-01
3	4.029155e-01	7.611662e-01	5.970845e-01	2.285714e-01
4	8.635235e-01	9.454094e-01	1.364765e-01	2.285714e-01
5	9.688054e-01	9.875222e-01	3.119462e-02	2.285714e-01
6	9.928698e-01	9.971479e-01	7.130199e-03	2.285714e-01
7	9.983702e-01	9.993481e-01	1.629760e-03	2.285714e-01
8	9.996275e-01	9.998510e-01	3.725165e-04	2.285714e-01
9	9.999149e-01	9.999659e-01	8.514663e-05	2.285714e-01
10	9.999805e-01	9.999922e-01	1.946209e-05	2.285714e-01
11	9.999956e-01	9.999982e-01	4.448477e-06	2.285714e-01
12	9.999990e-01	9.999996e-01	1.016795e-06	2.285714e-01
13	9.999998e-01	9.999999e-01	2.324102e-07	2.285714e-01
14	9.999999e-01	1.000000e-00	5.312234e-08	2.285714e-01
15	1.000000e-00	1.000000e-00	1.214225e-08	2.285714e-01
16	1.000000e-00	1.000000e-00	2.775371e-09	2.285714e-01
17	1.000000e-00	1.000000e-00	6.343704e-10	2.285714e-01
18	1.000000e-00	1.000000e-00	1.449989e-10	2.285713e-01
19	1.000000e-00	1.000000e-00	3.314249e-11	2.285706e-01
20	1.000000e-00	1.000000e-00	7.575385e-12	2.285702e-01
21	1.000000e-00	1.000000e-00	1.731504e-12	2.285698e-01
22	1.000000e-00	1.000000e-00	3.956835e-13	2.285201e-01
23	1.000000e-00	1.000000e-00	9.037215e-14	2.283951e-01
24	1.000000e-00	1.000000e-00	2.065015e-14	2.285012e-01
25	1.000000e-00	1.000000e-00	4.551914e-15	2.204301e-01

Gauß-Seidel method

Model problem:

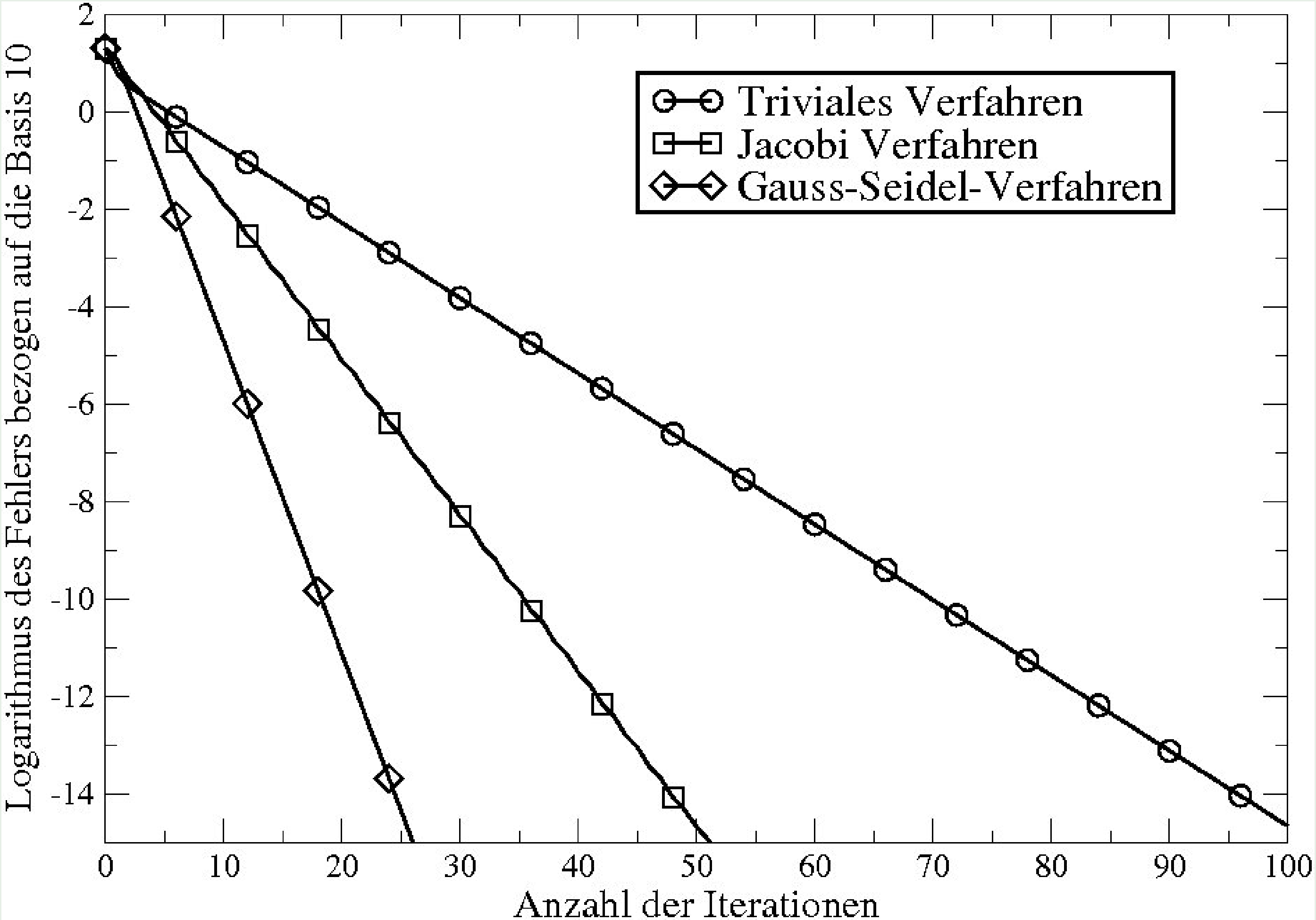

Abbildung: Convergence history $\log_{10} \varepsilon_m$ of the Gauß-Seidel method

Summary:

Splitting methods:

- Easy to derive
- Simple to implement

Rate of convergence:

- Is determined by $\rho(M) = \rho(B^{-1}(B - A))$.
- Rule of thumb: Choose the approximation B as close as possible w.r.t. A to obtain a good convergence behaviour.

Summary:

Typs of Splitting methods

- Trivial scheme: $B = I$ (bad rate of convergence)
- Jacobi method : $B = D$
- Gauß-Seidel method : $B = D + L$