Exercises for Iterative Solvers for Linear Systems

Exercises 1: (Splitting methods)
Consider the linear system of equations

$$\begin{pmatrix} a & 14 \\ 7 & 50 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{with} \quad a \in \mathbb{R}^+ \setminus \left\{ \frac{49}{25} \right\}.$$

(a) By means of the template

myjacobi.m

write a program to solve the above linear system of equations using the Jacobi method.

Study the convergence behaviour for the sequence of parameters $a = 1/2, 1, 2, 10, 100$ and arbitrary initial guess $\boldsymbol{x}_0 \in \mathbb{R}^2$ by calling the program

Aufgabe1_Jac.m .

Hint: The variation of both the parameter $a \in \mathbb{R}$ and the initial guess $\boldsymbol{x}_0 \in \mathbb{R}^2$ should be realized within the template **Aufgabe1_Jac.m**.

(b) By means of the template

mygs.m

write a program to solve the above linear system of equations using the Gauss-Seidel method.

Study the convergence behaviour for the sequence of parameters $a = 1/2, 1, 2, 10, 100$ and arbitrary initial guess $\boldsymbol{x}_0 \in \mathbb{R}^2$ by calling the program

Aufgabe1_GS.m .

Hint: The variation of both the parameter $a \in \mathbb{R}$ and the initial guess $\boldsymbol{x}_0 \in \mathbb{R}^2$ should be realized within the template **Aufgabe1_GS.m**.

(c) Compare the convergence behaviour of the Jacobi method and the Gauss-Seidel scheme for different parameter values by means of calling the template

Aufgabe1_Jac_GS.m

Hint: The variation of both the parameter $a \in \mathbb{R}$ and the initial guess $\boldsymbol{x}_0 \in \mathbb{R}^2$ should be realized within the template **Aufgabe1_Jac_GS.m**.

Exercise 2: (Splitting methods)
Consider the linear system of equations

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

The spectral radius for the iteration matrices of the Jacobi method and the Gauß-Seidel relaxation method are

$$\rho\left(M_J\right) = \sqrt{41/100}, \quad \rho\left(M_{GS}\right) = 41/100,$$

respectively. Furthermore, the matrix is consistency ordered. Calculate the optimal relaxation parameter for the Gauß-Seidel method using the above given informations.
By means of the template

mysor.m.

write a program to solve the above linear system of equations using the SOR method method.
Study the convergence behaviour with respect to the relaxation parameter $0 < \omega < 2$, whereby the initial guess is set to be $\boldsymbol{x}_0 = \mathbf{0} \in I\!R^3$. Therefore, call the template

Aufgabe2.m .

Hint: The variation of the relaxation parameter $0 < \omega < 2$ should be realized within the template **Aufgabe2.m**. Furthermore, the template visualizes automatically the spectral radius as a function of the relaxation parameter. Compare the location of the minimum of this function with your analytical result concerning the optimal parameter.

Exercises 3: (Method of steepest descent, CG method)
Let us consider the linear system of equations

$$\boldsymbol{A} = \begin{pmatrix} 168 & 24 & 338 & 27 & 27 & 53 & -7 & 80 \\ 24 & 178 & 169 & 72 & 53 & -103 & 17 & 80 \\ 338 & 169 & 1177 & 192 & -62 & -108 & -48 & 180 \\ 27 & 72 & 192 & 125 & 2 & -24 & 36 & 180 \\ 27 & 53 & -62 & 2 & 222 & 70 & 46 & 100 \\ 53 & -103 & -108 & -24 & 70 & 178 & 34 & 100 \\ -7 & 17 & -48 & 36 & 46 & 34 & 34 & 100 \\ 80 & 80 & 180 & 180 & 100 & 100 & 100 & 400 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 229 \\ 129 \\ 790 \\ -214 \\ 106 \\ -276 \\ -216 \\ -600 \end{pmatrix}$$

and

$$\boldsymbol{A} = \begin{pmatrix} 71 & -1 & 23 & -36 & 70 & 60 \\ -1 & 161 & 102 & 33 & 40 & -120 \\ 23 & 102 & 135 & 18 & 110 & -60 \\ -36 & 33 & 18 & 35 & -22 & -60 \\ 70 & 40 & 110 & -22 & 148 & 24 \\ 60 & -120 & -60 & -60 & 24 & 144 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 187 \\ 194 \\ 360 \\ -45 \\ 432 \\ 0 \end{pmatrix}.$$

(a) Write a program for the method of steepest descent using the template

mygraddesc.m.

Study the convergence history by calling the template

Aufgabe3_Descent.m .

Hint: The determination of the linear system to be used takes place automatically within the template **Aufgabe3_Descent.m**. The definition of the maximal iteration number as well as the stopping criterion are also done within **Aufgabe3_Descent.m**.

(b) Write a program for the method of conjugate gradients using the template

mycg.m.

Study the convergence history by calling the template

Aufgabe3_CG.m .

Hint: The determination of the linear system to be used takes place automatically within the template **Aufgabe3_CG.m**. The definition of the maximal iteration number as well as the stopping criterion are also done within **Aufgabe3_CG.m**.

(c) Compare the convergence behaviour of both methods for different parameter values (stopping criterion and maximum number of iterations) by means of the template

Aufgabe3_Descent_CG.m .

Hint: The determination of the linear system to be used takes place automatically within the template **Aufgabe3_Descent_CG.m**. The definition of the maximal iteration number as well as the stopping criterion are also done within **Aufgabe3_Descent_CG.m**.

Exercises 4: (Method of conjugate gradients)
Consider the boundary value problem

$$-u_{xx} - u_{yy} - au_x - bu = 0 \ \text{ on } \ \Omega = (0,1) \times (0,1)$$

with

$$u = 0 \ \text{ on } \ \partial\Omega.$$

The domain $\overline{\Omega}$ will be discretized due to $(x_i, y_j) = (i \cdot h, j \cdot h)$ with $h = 1/100$, $i, j = 0, \ldots, 100$. For the approximation of the derivative we introduce both central and one-sided differences like

$$\begin{aligned}
u_{xx}(x_i, y_j) &\approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \\
u_{yy}(x_i, y_j) &\approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} \\
u_x(x_i, y_j) &\approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}.
\end{aligned}$$

Investigate the convergence history ($\|\boldsymbol{r}_n\|_2$ versus n) of the method of conjugate gradients in accordance with

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0} \in \mathbb{R}^N, \ \text{ with } \ N = 99^2$$

and

$$\boldsymbol{u} = (u_{1,1}, \ldots, u_{1,N}, u_{2,1}, \ldots, u_{2,N}, \ldots, u_{N,N})^T$$

with respect to different parameter values $a, b \in \mathbb{R}_0^+$ and a fixed initial guess $\boldsymbol{u}_0 = (1, \ldots, 1)^T$. Therefore, please use the template

mycg.m

developed within the exercise considered above. Introduce for example

$$(a,b) = (0,0), (a,b) = (0,10), (a,b) = (10,0), (a,b) = (10,10), \quad \text{and so on.}$$

Discuss the properties of the matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ depending on the choice of the parameters a and b?
Hint: The determination of the linear system to be used takes place automatically within the template **Aufgabe4.m**. The definition of the parameter set $a, b \in \mathbb{R}_0^+$ is realized within **Aufgabe4.m**.

Exercises 5: (Multigrid method)
Within the template

Aufgabe5.m.

you will find a realization of a two grid method with respect to the one-dimensional Poisson's equation

$$u''(x) = k^2\pi^2 \sin(k\pi x)$$

using the damped Jacobi scheme as a smoother as well as the injection and linear prolongation for the remaining parts of the coarse grid correction step.
Investigate the convergence behaviour for

(a) different relaxation parameters ω,

(b) different combination of numbers for the pre- and post-smoothing steps, ν_1, ν_2 respectively,

(c) different frequencies k within the right hand side $k^2\pi^2 \sin(k\pi x)$ of the ordinary differential equation. Thereby, take a specific attention on the grid spacing defined by the choice of the value l, which determines the number of inner points to be $2^{(l+1)} - 1$.

Exercises 6: (Krylov subspace methods)
Let us once again consider both the partial differential equation and the linear system of equations defined in exercise 4. The discretization is realized within each program named in the form **Aufgabe6_name.m**.

(a) Write a program for the BiCG method using the template

mybicg.m.

Study the convergence history by calling the template

Aufgabe6_BICG.m .

Hint: The definition of the parameter $a, b \in \mathbb{R}_0^+$ is realized within the template **Aufgabe6_BICG.m**.

(b) Write a program for the CGS method using the template

mycgs.m.

Study the convergence history in comparison to the BiCG method by calling the template

Aufgabe6_BICG_CGS.m .

Hint: The definition of the parameters $a, b \in \mathbb{R}_0^+$ is realized within the template **Aufgabe6_BICG_CGS.m**.

(c) Write a program for the BiCGSTAB method using the template

mybicgstab.m.

Study the convergence history in comparison to the BiCG and the CGS method by calling the template

Aufgabe6_BICG_CGS_BICGSTAB.m .

Hint: The definition of the parameter $a, b \in \mathbb{R}_0^+$ is realized within the template **Aufgabe6_BICG_CGS_BICGSTAB.m**.

(d) Study the convergence history of the methods BiCG, CGS, BiCGSTAB and GMRES by calling the template

Aufgabe6_BICG_CGS_BICGSTAB_GMRES.m.

Hint: The definition of the parameters $a, b \in \mathbb{R}_0^+$ is realized within the template **Aufgabe6_BICG_CGS_BICGSTAB_GMRES.m**.

(e) Study the convergence history of the methods BiCG, CGS, BiCGSTAB, GMRES, GMRES(m) for $m = 25, 50, 75, 100$ by calling the template

Aufgabe6_BICG_CGS_BICGSTAB_GMRESm.m.

Hint: The definition of the parameters $a, b \in \mathbb{R}_0^+$ is realized within the template **Aufgabe6_BICG_CGS_BICGSTAB_GMRESm.m**.